Belinskaya's theorem is optimal

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I/ Ergodic theory

(X,μ) standard proba space $\left(\cong([0,1],\mathrm{Leb})\right)$

All sets & maps are measurable. All is up to measure zero.

$$\mathrm{Aut}(X,\mu) := \{T : X \to X \text{ bij } \forall A \subseteq X,\ \mu(A) = \mu(T(A))\}/\sim_\mu$$

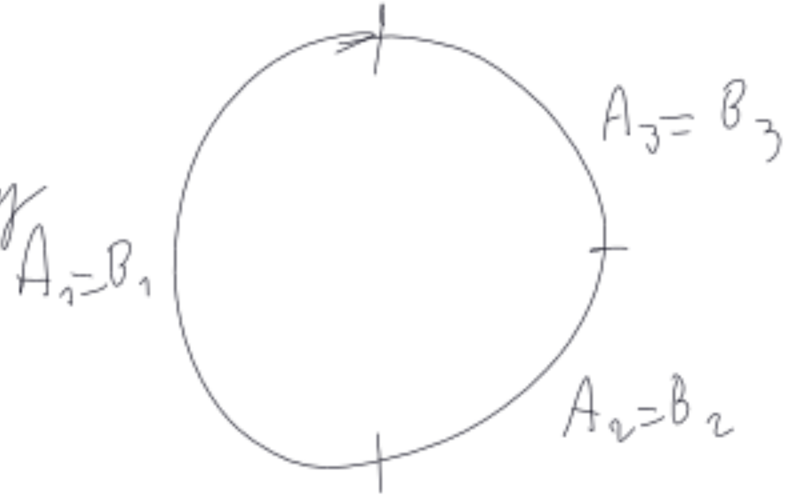
the group of MP transformations

$T_1 \sim_\mu T_2$ if $\mu(\{x \in X : T_1(x) = T_2(x)\}) = 1$

Ex: • Irrational rotation $\alpha \in \mathbb{R}\setminus\mathbb{Q}$

$X = [0,1)$, $\mu = \text{leb}$ — $\boxed{\text{not}}$ weakly mixing

$T_\alpha(x) = x + \alpha \mod 1$



• Bernoulli shifts: $X = \{0,1\}^{\mathbb{Z}}$

$$\mu = \left(\frac{1}{2}(\delta_0 + \delta_1)\right)^{\otimes \mathbb{Z}}$$

$$T((x_n)_{n\in\mathbb{Z}}) = (x_{n-1})_{n\in\mathbb{Z}}$$

Mixing: $\mu(T^n(A)\cap B) \to \mu(A)\mu(B)$ as $n \to +\infty$

$\forall A, B \subseteq X$

Both are ergodic: $\forall A \subseteq X$, $T(A) = A \Rightarrow \mu(A) = 0$ or 1

T is ergodic if

T is weak mixing: $\forall A_1, \dots, A_k, B_1, \dots, B_k$ $\forall \varepsilon > 0$

$\exists n \,/\, \forall i = 1 \dots k$ $\mu(T^n(A_i)\cap B_i) \underset{\varepsilon}{\simeq} \mu(A_i)\mu(B_i)$

Def: $T_1, T_2 \in \mathrm{Aut}(X,\mu)$ are <u>conjugate</u> if $\exists\, S \in \mathrm{Aut}(X,\mu)$

$$S T_1 S^{-1} = T_2$$

T_α is conjugate to $T_{-\alpha}$ via $S(x) = 1-x$

<u>Thm</u> (Halmos von Neumann). T_α is conjugate to T_β iff $\alpha = \pm\beta \mod 1$

II) Full groups & Belinskaya's theorem

Def: $T \in \mathrm{Aut}(X,\mu)$, its <u>full group</u> $[T]$ is

$$[T] := \{ U \in \mathrm{Aut}(X,\mu) : \forall x \in X,\ U(x) \in \mathrm{Orb}_T(x) \}$$

This is a Polish group, natural complete biinvariant metric (unif. metric)

$$d_u(T_1, T_2) = \mu(\{x \in X : T_1(x) \neq T_2(x)\})$$

<u>Thm</u> (Dye '59, Belinskaya '69)

$\forall T_1, T_2 \in Aut(X,\mu)$ <u>ergodic</u> $\exists S \in Aut(X,\mu)$

$\forall x$, $\underline{S(Orb_{T_1}(x)) = Orb_{T_2}(S(x))}$

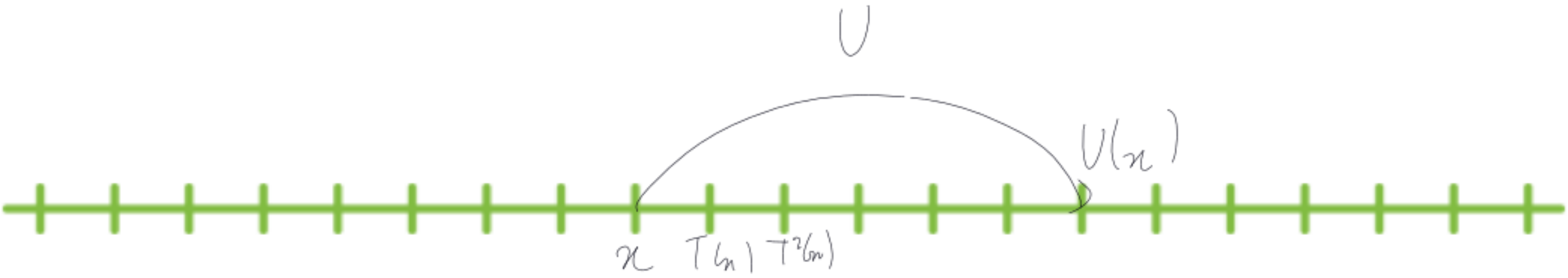
$\Longleftrightarrow S[T_1]S^{-1} = [T_2]$

In other words, T_1 is conjugate to an element of $[T_2]$ which has same orbits as T_2

Def: $T \in Aut(X,\mu)$ ergodic ($\Rightarrow T$ has ∞ orbits)

For $V \in [T]$, its <u>cocycle</u> is $c_V : X \to \mathbb{Z}$ uniquely def by:

$$V(x) = T^{c_V(x)}(x)$$



$c_U(x) = 6$

Thm (Belinskaya '69) T ergodic, $U \in [T]$ with same orbits as T.

suppose $\displaystyle\int_X |c_U(x)| \, d\mu(x) < +\infty$

then T and U are flip-conjugate: T is conj to $U^{\pm 1}$

Thm (Carderi, Joseph, LM, Tessera)

Let $\varphi : \mathbb{N} \to \mathbb{R}^+$ st $\displaystyle\lim_{n \to +\infty} \frac{\varphi(n)}{n} = 0$ (φ is sublinear)

$\forall T$ ergodic, $\exists U \in [T]$ with same orbits as T, not flip conj to T, but $\displaystyle\int_X \varphi(|c_U(x)|) \, d\mu(x) < +\infty$

ex: $\varphi(n) = \frac{n}{\log(n+2)}$

$\rightsquigarrow c_U \in L^p \ \forall p < 1$

Key tool: φ- integrable full groups

Given φ as before define the "φ-integrable full group" of $T \in \mathrm{Aut}(X,\mu)$ ergo

lug: $$[T]_\varphi := \left\{ U \in [T] : \int_X \varphi(|c_U(x)|)\, d\mu(x) < +\infty \right\}$$

Assume: • φ is subadditive: $\varphi(n+m) \leq \varphi(n) + \varphi(m)$

• φ is non-decreasing

• φ is Hausdorff: $\varphi(0) = 0$, $\varphi(n) > 0 \quad \forall n \neq 0$

Then $[T]_\varphi$ is a group: $c_{U_1U_2}(x) = c_{U_1}(U_2(x)) + c_{U_2}(x)$

$$\int \varphi(|c_{U_1U_2}(x)|) \leq \int_X \varphi|c_{U_1}(U_2(x))| + \int \varphi|c_{U_2}(x)|$$

Fact: $[T]_\varphi$ is a Polish group

metric: $d_\varphi(V_1, V_2) = \int_X \varphi\left(|c_{V_1}(x) - c_{V_2}(x)|\right) d\mu(x)$

complete, right-inv, separable

$\rightsquigarrow$ Can do Baire category arguments!

$APER = \{T \in Aut(X,\mu) : T$ has only ∞ orbits$\}$ is d_μ- closed
(aperiodic)
(in part $APER \cap [T]_\varphi$ is closed)

Thm: $\{V \in APER \cap [T]_\varphi$: V has same orbits as T & V is weak mixing$\}$ is dense G_δ

φ sublinear
T ergodic $\rightsquigarrow$ Main Thm when T is not WMIX

$MAlg(X,\mu) = \{A \subseteq X$ Borel$\}/\sim_\mu$
Polish $d_\mu(A,B) = \mu(A \Delta B)$

Thm (Conze) T ergodic $\{A \in \mathrm{MAlg}(X,\mu) : (T_A)_{\restriction A} \text{ is WMIX}\}$ is dense G_δ

induced trsfo: $A \subseteq X$, $\forall^* x \in A$, $\exists n \geq 1$ / $T^n(x) \in A$

$$\underset{\mathrm{Aut}(A)}{\overset{\cap}{T_A(x)}} = \begin{cases} T^n(x), & n = \min\{m : T^m(x) \in A\} \\ \end{cases}$$